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Radiation Forces in Inhomogeneous Media

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Force terms arising from radiation impinging on spatial and time inhomogeneities of a medium can frequently be represented by a simple partial differentiation operating on only the structural elements of the Lagrangian. These power and force density terms are energy and momentum sinks (or sources) of the raidation field. They accordingly occur as right-hand members in the usual divergence expressions for energy and momentum. The method of description is illustrated for acoustic radiation, for electromagnetic radiation in a material medium, and for electromagnetic radiation in a gravitational field. A coherent discussion of different kinds of radiation requires a reevaluation of the concept of wave momentum. The particle-momentum concept is implicit in the modified concept of wave momentum for "matter" waves. It then appears that the symmetry of the energy-momentum tensor only holds for electromagnetic radiation in free space and for matter waves associated with free and noninteracting particles.

1. INTRODUCTION

THE energy-momentum conservation in a continuous medium is usually given by a divergence equation of the form

$$\partial_\lambda \mathfrak{T}_\nu{}^\lambda = 0, \quad \lambda, \nu = 0, 1, 2, 3, \tag{1}$$

where $\mathfrak{T}_\nu{}^\lambda$ is known as the energy-momentum tensor whose elements may be identified according to the scheme

$$\begin{aligned} &\mathfrak{T}_0{}^0 = \text{energy density}, \\ &\mathfrak{T}_0{}^1, \mathfrak{T}_0{}^2, \mathfrak{T}_0{}^3 = \text{energy flow vector in space}, \\ &\mathfrak{T}_1{}^0, \mathfrak{T}_2{}^0, \mathfrak{T}_3{}^0 = \text{momentum density vector in space}, \\ &\begin{Bmatrix} \mathfrak{T}_1{}^1 & \mathfrak{T}_1{}^2 & \mathfrak{T}_1{}^3 \\ \mathfrak{T}_2{}^1 & \mathfrak{T}_2{}^2 & \mathfrak{T}_2{}^3 \\ \mathfrak{T}_3{}^1 & \mathfrak{T}_3{}^2 & \mathfrak{T}_3{}^3 \end{Bmatrix} = \text{spatial stress components}. \end{aligned} \tag{2}$$

If a system satisfies Eq. (1), we call it "closed," for example, a closed system of electromagnetic or acoustic radiation. The term also applies to the energy and momentum associated with matter provided that the interaction with the gravitational field is negligible.

Actual physical systems, however, are rarely closed in the sense that energy and momentum of a particular form are conserved. In point of fact the more interesting physical situation usually arises if there is some form of energy and momentum exchange between systems. The interaction of a particular system with extraneous sources (or sinks) of energy and momentum may formally be represented by a nonvanishing divergence of the form

$$\partial_\lambda \mathfrak{T}_\nu{}^\lambda = \mathfrak{f}_\nu. \tag{3}$$

The four-vector-like quantity $\mathfrak{f}_\nu$ on the right-hand side can physically be identified as follows:

$$\mathfrak{f}_0 = \text{a density of power sinks (or sources) of the radiation field}, \tag{3a}$$

$$\{\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3\} = \text{a radiation-force density}. \tag{3b}$$

Extensions of the physical interpretation of (3) will be a major concern in this article. We may set the stage for this discussion by first recalling some well-known cases where $\mathfrak{f}_\nu$ has an accepted interpretation.

For electromagnetic waves impinging on a field of electric charges and currents one knows that

$$\mathfrak{f}_\nu = F_{\nu\lambda}\mathfrak{c}^\lambda, \tag{4a}$$

where $F_{\nu\lambda}$ is the electromagnetic field (E,B) and $\mathfrak{c}^\lambda$ the four-vector of charge and current density. The spatial part of (4a) is the well-known Lorentz force.

A more difficult situation arises if $\mathfrak{c}^\lambda$ represents only the macroscopic conduction current in a material medium. What happens to the Lorentz forces associated with the (not directly observable) microphysical current components in the medium? We intend to deal with this question in Sec. 3, where it will appear that an observable residue of (4a) remains if the medium has a dielectric or magnetic nonuniformity. Another reasonably well-established case, where $\mathfrak{f}_\nu \neq 0$, is encountered in general relativity. The quantity $\mathfrak{f}_\nu$ is then given by the expression

$$\mathfrak{f}_\nu = \mathfrak{T}_\kappa{}^\lambda \Gamma_{\lambda\nu}{}^\kappa, \tag{4b}$$

where $\Gamma_{\lambda\nu}{}^\kappa$ are the Christoffel expressions of the linear connection of the space-time manifold. The right-hand member of (4b) in the Newtonian approximation becomes

$$\mathfrak{T}_\kappa{}^\lambda \Gamma_{\lambda\nu}{}^\kappa \to \rho X_\nu \quad \nu = 1, 2, 3 \tag{5}$$

with ρ the mass density of matter and $X_\nu = (X_1, X_2, X_3)$ the spatial vector of gravitational acceleration.

It is of some interest to note here that the expression (4b) is generated by a nonuniformity of the medium (non-Euclidean properties of the space-time manifold). It will, in general, appear to be a fruitful point of view to consider $\mathfrak{f}_\nu$ as intimately related to the nonuniformities of the medium, particularly when the actual physical nonuniformities of the medium (e.g., dielectric nonuniformities) prevail over the commonly small influence of the space-time nonuniformities of general relativity.

An application of the formula (3) to an arbitrary radiation field will, as a rule, lead to few difficulties so long as we restrict ourselves to the first equation for $\nu=0$. The energy density and the energy flow are usually well defined concepts. The "spatial" equation $\nu=1, 2, 3$, however, may invoke uncertainties. The questions arises: what are the momentum and stress components of a radiation field? Again, the answer is reasonably well agreed upon for electromagnetic radiation in free space: $\mathfrak{T}_\nu{}^\lambda$ (λ, $\nu=1, 2, 3$) are the Maxwell stresses and $\mathfrak{T}_\nu{}^0$ ($\nu=1, 2, 3$) is the electromagnetic momentum which equals $1/c^2$ times the energy flow $\mathfrak{T}_0{}^\lambda$ ($\lambda=1, 2, 3$), or in magnitude $1/c$ times the energy density $\mathfrak{T}_0{}^0$, with c the free-space velocity of light.

The same question for electromagnetic waves in an arbitrary material medium will lead to rather serious difficulties, associated with the fact that we now have to answer the question of what happens to the Lorentz forces on the microphysical (bound) charges and currents in the medium. It appears that the previous definition of wave momentum has here no direct physical meaning.

Similarly, a wave-momentum definition of $1/c$ times the energy density $\mathcal{E}$ of radiation is not of much help in acoustics, neither is the definition $\mathcal{E}g/c^2$ with g the group velocity.

It is the aim of the next section to show that the more generally applicable relativistic definition of wave momentum is

$$\text{wave momentum}=\frac{\text{wave energy}}{\text{phase velocity of wave motion}}. \tag{6}$$

The old relativistic definition,

$$\text{wave momentum}=\frac{\text{wave energy}\times\text{group velocity}}{(\text{free space light velocity})^2}, \tag{6a}$$

will appear to have a precise physical meaning, only if the product of group velocity g and phase velocity u equals the square of the free-space light velocity c.

Armed with the new definition of wave momentum (6), one can re-assess the problem of radiation forces and stresses, because it is the change of this wave momentum that produces the radiation forces on nonuniformities in a medium. [See Eq. (3).] The old definition (6a) in general does not admit of such a relation.

2. WAVE MOMENTUM VERSUS PARTICLE MOMENTUM

The definition (6) of wave momentum was probably first given by Poynting prior to the era of relativity. It has, however, survived in spite of its seemingly nonrelativistic character for the case that $ug\neq c^2$. Its usefulness has been demonstrated in a number of investigations on acoustic streaming. Of the many possible references I may mention Eckart.[1] Piercy[2] showed that good reproducible data on sound absorption can be obtained on the basis of the definition (6) of wave momentum. Weinreich *et al.*[3] have also used the concept in an investigation of the acoustoelectric effect. Sturrock[4,5] and the author[6] have independently given a discussion which may be considered as an attempt to reinstate the relativistic respectability of Poynting's wave momentum.

It thus appears that the somewhat too formal relativistic argument underlying the definition (6a) can be misleading. Keeping closely to first principles, I may therefore attempt a direct and simple physical discussion for the justification of definition (6).

Let us consider an energy-momentum vector with components

$$p_\lambda=\{p_0,p_1\}=\{E,p\}, \quad \lambda=0, 1. \tag{7}$$

For simplicity we take only one spatial component. Let

$$t'=[t+(v/c^2)x]\beta, \quad x'=(vt+x)\beta, \qquad \beta=\frac{1}{(1-v^2/c^2)^{1/2}} \tag{8}$$

be a Lorentz transformation associated with a translation in the x direction with velocity v. The Jacobian matrices of (8) are

$$A_\lambda{}^{\lambda'}=\begin{pmatrix} A_0{}^{0'} & A_1{}^{0'} \\ A_0{}^{1'} & A_1{}^{1'}\end{pmatrix}=\begin{pmatrix} 1 & v/c^2 \\ v & 1\end{pmatrix}\beta, \tag{8a}$$

and the inverse

$$A_{\lambda'}{}^{\lambda}=\begin{pmatrix} A_{0'}{}^{0} & A_{1'}{}^{0} \\ A_{0'}{}^{1} & A_{1'}{}^{1}\end{pmatrix}=\begin{pmatrix} 1 & -v/c^2 \\ -v & 1\end{pmatrix}\beta. \tag{8b}$$

From Hamiltonian mechanics we may extract the information that energy and momentum constitute the components of an intrinsically covariant vector. Its transformation is thus given by

$$p_{\lambda'}=A_{\lambda'}{}^{\lambda}p_\lambda, \tag{9}$$

or expanded in the components given by (7) and (8b)

$$E'=(E+vp)\beta, \quad p'=[(v/c^2)E+p]\beta. \tag{9a}$$

Case I (*particle momentum*). Suppose E and p represent the energy and momentum of a particle with rest mass m. From Galilean-Newtonian mechanics we know that momentum should transform according to the equation

$$p'=p+mv. \tag{10}$$

A compatibility of (9a) and (10) for $v\ll c$ is possible if we reassess our concept of energy by postulating the well-known relation

$$E=mc^2.$$

[1] C. Eckart, Phys. Rev. **73**, 68 (1948).

[2] J. E. Piercy, J. Acoust. Soc. Am. **29**, 770 (1957). Also thesis University of London, 1953 (unpublished).

[3] G. Weinreich, Phys. Rev. **107**, 317 (1957).

[4] P. A. Sturrock, J. Appl. Phys. **31**, 2052 (1960).

[5] P. A. Sturrock, Phys. Rev. **121**, 18 (1961).

[6] E. J. Post, Phys. Rev. **118**, 1113 (1960).

Case II (*wave momentum*). Let us now suppose that E and p represent the energy and momentum of a quantum of radiation associated with an acoustic or electromagnetic wave of phase velocity u (photon or phonon—no rest-mass energy). Dividing the equations (9a) by $\hbar$ one obtains the following transformation equations for the frequency and wave number of the quantum of wave motion:

$$\omega' = (\omega + vk)\beta,$$
$$k' = (\omega v/c^2 + k)\beta. \tag{9b}$$

It thus follows from the contragradient relation between (8) and (9b) that the eikonal $(\omega t - kx)$ is a Lorentz invariant for any wave motion. The closest Galilean result with which (9b) can be compared is the ordinary Doppler formula

$$\omega' = \omega(1 + v/u) \tag{11}$$

for the moving observer.

The compatibility of (11), (9a), and (9b) suggests the introduction of a wave momentum

$$p = E/u, \tag{6}$$

which is Poynting's definition of wave momentum (6).

The difference between Case I and Case II is that Case I, having a well-defined concept of momentum, requires a reconsideration of the energy concept, while Case II, having a well-defined concept of energy requires a reconsideration of the momentum concept. Hence, the wave momentum thus obtained is not the commonly accepted momentum (6a) but rather the momentum defined by Poynting (6) where u is the phase velocity in the medium under consideration. The following examples of limiting cases may be helpful to illustrate the nature of (6) and the fact that the special definition (6a) is contained in (6).

A. Electromagnetic Waves in Free Space

$$u = c \tag{12}$$

hence

$$p = E/c. \tag{13}$$

The equations (9b) become

$$\omega' = \omega(1 + v/c)\beta,$$
$$k' = k(1 + v/c)\beta. \tag{14}$$

Thus, reproducing the well-known result that ω and k transform in the same manner, so that ω'/k' also equals c.

B. The Wave Function of a Freely Moving Particle

Let the rest mass of the particle be m_0 and its (group) velocity g. In the geometric optical approximation one can speak of a phase velocity u of the wave function of the particle given by the ratio

$$u = E/p = \omega/k = c^2/g, \tag{15}$$

provided the energy of the particle is meant to include the rest-mass energy according to

$$E = m_0 c^2/(1 - g^2/c^2)^{1/2} = mc^2. \tag{16}$$

Hence, the wave momentum associated with the matter wave is according to definition (6)

$$p = E/u = m_0 g/(1 - g^2/c^2)^{1/2}, \tag{17}$$

which is equivalent to the relativistic particle momentum, because formulas (15) and (6) are identical. The equivalence of (15) and (6) follows from the fact that the expectation value of the particle velocity equals the group velocity of the wave packet associated with the particle, provided conditions prevail that justify the geometric optical approximations. Of course, the same result would have been obtained with definition (6a).

C. Particle Moving in a Potential

An interesting difference between the two definitions of wave momentum occurs if the particle moves in a potential field φ, where φ is assumed to be a spatial scalar. Suppose that the field φ is sufficiently smooth to admit a geometric optical approximation that maintains the validity of (15). Hence $E = mc^2 + \varphi$ now yields the phase velocity

$$u = E/p = (mc^2 + \varphi)/mg. \tag{15a}$$

The wave-momentum definition (6) still contains the classical inertia momentum of the particle.

The definition (6a), however, gives

$$p = mg + \varphi g/c^2$$

where the extra term has the form of a "kinetic" momentum. Hence, contrary to our initial assumption that the particle moves in a scalar field we suddenly find a vector-potential extension $\varphi g/c^2$ which can be thought of as having been generated from φ by a Lorentz transformation. Clearly, the applied field should be independent of the motion of the particle. Hence, here is a case in point where definition (6a) must be discarded, while definition (6) ascertains confluence with well-established facts.

It should be noted that the discussion in this section hinges critically on the assumption that energy and momentum are the components of an intrinsically *covariant* vector.

Conclusions equivalent to those just obtained can also be derived directly from a discussion of the transformation of the energy-momentum tensor itself. It is then essential to consider the energy-momentum tensor as a tensor with mixed co- and contra-variant transformation behavior, as expressed in Eqs. (1) and (2). For a compatibility check one can then compare with classical expressions for the transformation of the energy flow.

A direct relation between the energy-momentum vector and the energy-momentum tensor can be established in those cases where the energy-momentum tensor may be considered as the direct product of the flow vector of "particle" density $\mathfrak{N}^\lambda$ and the energy momentum vector p_ν. The condition implies the existence of a system of noninteracting particles moving with uniform drift velocity g. The energy-momentum tensor is, as it should be, of mixed transformation behavior and can be written in the form

$$\mathfrak{T}_\nu{}^\lambda = p_\nu \mathfrak{N}^\lambda. \tag{18}$$

Expressed in the energy density $\mathcal{E}$, the group velocity g and the phase velocity u, the spatially one-dimensional form of the energy-momentum tensor thus becomes

$$\mathfrak{T}_\nu{}^\lambda = \begin{pmatrix} \mathfrak{T}_0{}^0 & \mathfrak{T}_0{}^1 \\ \mathfrak{T}_1{}^0 & \mathfrak{T}_1{}^1 \end{pmatrix} = \begin{pmatrix} \mathcal{E} & \mathcal{E}g \\ -\mathcal{E}/u & -\mathcal{E}g/u \end{pmatrix}. \tag{18a}$$

Symmetry considerations as an invariant feature of the energy-momentum tensor only apply to a modified form that has no mixed transformation behavior. A completely covariant tensor can be obtained by means of the metric tensor

$$\mathfrak{T}_{\nu\lambda} = \mathfrak{T}_\nu{}^\kappa g_{\kappa\lambda}, \tag{19}$$

with

$$g_{\kappa\lambda} = \begin{pmatrix} c^2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}$$

The expansion of (19) with the use of (18a) and (20) yields

$$\mathfrak{T}_{\nu\lambda} = \begin{pmatrix} \mathcal{E}c^2 & -\mathcal{E}g \\ -\mathcal{E}c^2/u & \mathcal{E}g/u \end{pmatrix}. \tag{19b}$$

It follows from (19b) that

$$\mathfrak{T}_{\nu\lambda} = \mathfrak{T}_{\lambda\nu} \tag{21}$$

if, and only if

$$ug = c^2. \tag{21a}$$

The condition (21a) is met for electromagnetic radiation in free space as well as for the energy-momentum tensor that can be constructed from the single-particle wave functions of the noninteracting system described by (18). As can be seen from (15a), any form of interaction will upset the condition (21a).

Hence, symmetry is also not a general property of the energy-momentum tensor of electromagnetic radiation in a material medium. A spatial asymmetry exists if the spatial vectors $\mathbf{k}$ and $\mathbf{g}$ are not parallel, as in anisotropic media. The radiative torque associated with this asymmetry has been demonstrated quantitatively in a famous experiment. Beth[7] has shown that polarized light really exerts a torque density $\mathbf{E}\times\mathbf{D}$ in a dielectrically anisotropic crystal.

[7] R. Beth, Phys. Rev. **50**, 115 (1937).

A similar phenomenon should be expected for polarized acoustic waves propagating in an elastically anisotropic medium. An experimental verification, however, would be no minor endeavor.

In a recent paper, Sturrock[5] questions the meaning of a nonsymmetric energy-momentum tensor for a classical system. We hope the above considerations have clarified this point.

3. RADIATION FORCES ON INHOMOGENEITIES IN THE MEDIUM

Some examples of radiation forces on nonuniformities of the medium have been mentioned in the Introduction. They were cases satisfying the condition (21a) of the previous section. The emphasis in this section will be on the cases of acoustic radiation and on electromagnetic radiation in a material medium, neither of which meets the condition (21a).

The electromagnetic case may be approached from the point of view of the Lorentz forces (4a) acting on the microphysical charges and currents. I want to show that the macrophysical observable effect of the micro-Lorentz forces results in the usual contribution to polarization and magnetization, plus a residual macroforce which comes into play where the medium is nonuniform. The general line of argument is as follows:

Let

$$\partial_{[\kappa} f_{\lambda\nu]} = 0, \tag{22}$$

and

$$\partial_\nu \mathfrak{f}^{\lambda\nu} = \mathfrak{c}^\lambda$$

be the Maxwell-Lorentz equation of the microfields in a nonconducting medium. The field $\mathfrak{c}^\lambda$ represents bound micro charges and currents and $f_{\lambda\nu}$ and $\mathfrak{f}^{\lambda\nu}$ are assumed to be related via the metric tensor.

The fields $f_{\lambda\nu}$ and $\mathfrak{f}^{\lambda\nu}$ are unobservable when the matter is in a neutral macrostate. An external field will produce a deformation of the microfields manifesting itself as an observable macrofield.

Let the deformation be given by a one parameter infinitesimal Lie group with deformation vector u^ν, and let the Lie differential operator be denoted by $\mathfrak{L}$. The deformation then generates the observable components

$$\begin{aligned} &\mathfrak{L} f_{\lambda\nu} = F_{\lambda\nu}, \\ &\mathfrak{L}\mathfrak{f}^{\lambda\nu} = {}_0\mathfrak{G}^{\lambda\nu}, \\ &\mathfrak{L}\mathfrak{c}^\lambda = 2\partial_\nu(u^{[\nu}\mathfrak{c}^{\lambda]}) = -\partial_\nu \mathfrak{M}^{\lambda\nu}, \end{aligned} \tag{23}$$

where $\mathfrak{M}^{\lambda\nu}$ represents the polarization and magnetization while

$$\mathfrak{G}^{\lambda\nu} = {}_0\mathfrak{G}^{\lambda\nu} + \mathfrak{M}^{\lambda\nu} \tag{23a}$$

is the macromagnetic field and dielectric displacement. It follows from (22), (23), and (23a) that the macrofields satisfy the Maxwell equations

$$\partial_{[\kappa}F_{\lambda\nu]}=0\,, \tag{24}$$
$$\partial_\nu \mathfrak{G}^{\lambda\nu}=0\,, \quad \text{(no macrocurrent)}$$

because $\partial_\nu\mathfrak{L}=\mathfrak{L}\partial_\nu$.

The Maxwell-Lorentz form of the energy-momentum equation (3) of the microfield is

$$\partial_\lambda\{\mathfrak{l}\delta_\nu{}^\lambda - f_{\nu\sigma}\mathfrak{f}^{\lambda\sigma}\}=\mathfrak{c}^\sigma f_{\nu\sigma}\,, \tag{25}$$

where $\mathfrak{l}$ is the Lagrangian density of the microfield. The Lie-deformation of (25) retaining the observables (23) becomes

$$\partial_\lambda\{{}_0\mathfrak{L}\delta_\nu{}^\lambda - F_{\nu\sigma 0}\mathfrak{G}^{\lambda\sigma}\}=-F_{\nu\sigma}\partial_\lambda\mathfrak{M}\mathfrak{k}^{\sigma\lambda}\,. \tag{26}$$

Standard manipulations then show that (26) can be written in the form

$$\partial_\lambda\{\mathfrak{L}\delta_\nu{}^\lambda - F_{\nu\sigma}\mathfrak{G}^{\lambda\sigma}\}=\partial_{(\nu)1}\mathfrak{L} \tag{27}$$

with $\mathfrak{L}={}_0\mathfrak{L}+{}_1\mathfrak{L}$; ${}_1\mathfrak{L}$ being the part of the Lagrangian that is associated with the polarization and magnetization of the medium.

The right-hand member of (27) denotes a partial derivative $\partial_{(\lambda)}$, meant to operate only on the structural elements of $\mathfrak{L}$. It only differs from zero if there is an intrinsic space and time dependence of the dielectric and magnetic properties of the medium. Physically it represents that part of the micro-Lorentz forces that cannot be accommodated in the polarization and magnetization, thus leading to a net macroforce on the medium.

The physical meaning of the right-hand member of (27) has been previously discussed in Chap. IV, Sec. 3, of Ref. 8. The term is there derived directly from the phenomenological equations (24) by evading the commonly hidden assumption of medium uniformity.

It can also be shown [Chap. IX, Sec. 4, Eq. (9.57), Ref. 8] that

$$\partial_{(\lambda)0}\mathfrak{L}={}_0\mathfrak{T}_\nu{}^\kappa\Gamma_{\lambda\kappa}{}^\nu\,. \tag{28}$$

Hence the complete right-hand member of (27), when including nonuniformities of a gravitational origin, simply becomes

$$\mathfrak{f}_\lambda=\partial_{(\lambda)}\mathfrak{L}\,. \tag{29}$$

A discussion of acoustic radiation in a medium with changing density and elastic stiffness should lead to a result similar to (27), with a right-hand member of the form (29) giving the force density of radiation acting on nonuniformities in the medium. The derivation is really implicit in the discussions encountered in the introductory classical parts of many textbooks on field theory. The result is also stated in Eq. (13) of Ref. 6. The following example may be helpful as an illustration for the use of (29), because an explicit discussion was omitted in Ref. 6.

Let

$$\mathfrak{L}=\frac{1}{2}\left\{\rho\left(\frac{\partial\xi}{\partial t}\right)^2 - s\left(\frac{\partial\xi}{\partial x}\right)^2\right\} \tag{30}$$

be the Lagrangian density of an inhomogeneous one-dimensional medium; the mass density ρ and the stiffness s both are functions of x, while ξ denotes the displacement.

The structural derivative (29) of the Lagrangian (30) becomes

$$\frac{\partial\mathfrak{L}}{\partial(x)}=\frac{1}{2}\left\{\left(\frac{\partial\rho}{\partial x}\right)\left(\frac{\partial\xi}{\partial t}\right)^2-\left(\frac{\partial s}{\partial x}\right)\left(\frac{\partial\xi}{\partial x}\right)^2\right\}, \tag{31}$$

or

$$\frac{\partial\mathfrak{L}}{\partial(x)}=\frac{1}{2}\left\{\left(\frac{\partial\ \ln\rho}{\partial x}\right)\rho\left(\frac{\partial\xi}{\partial t}\right)^2-\left(\frac{\partial\ \ln s}{\partial x}\right)s\left(\frac{\partial\xi}{\partial x}\right)^2\right\}. \tag{31a}$$

A pure traveling wave situation without reflections can exist in an inhomogeneous medium provided the acoustic impedance

$$Z=(\rho s)^{1/2}=\text{constant}. \tag{32}$$

It then follows from (31a) after time averaging that

$$\mathfrak{f}_\nu=\frac{\partial\mathfrak{L}}{\partial(x)}=-\frac{\partial\ \ln(s/\rho)^{1/2}}{\partial x}\mathcal{E}\,, \tag{33}$$

where $\mathcal{E}$ is the energy density of the radiation. The result (33) shows that radiation forces can exist in a reflectionless medium. This point was demonstrated very neatly in an interesting experiment by Hertz and Mende[9] in 1939, showing that a radiation pressure occurs at the interface of two liquids with the same acoustic impedance but different propagation velocities of sound.

The force density at a discontinuity goes to infinity. However, the limit

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\frac{\partial\mathfrak{L}}{\partial(x)}dx \tag{34}$$

is finite.

Rewriting (31)

$$\frac{\partial\mathfrak{L}}{\partial(x)}=\frac{1}{2}\left\{\left(\frac{\partial\rho}{\partial x}\right)\left(\frac{\partial\xi}{\partial t}\right)^2+\frac{\partial s^{-1}}{\partial x}(T)^2\right\} \tag{35}$$

with the stress $T=s(\partial\xi/\partial x)$, one finds from (34) that the force per unit area on the boundary equals the difference of the energy densities at the interface. This follows because $\partial\xi/\partial t$ and T are continuous at the boundary.

4. CONCLUSION

The aim of the present article may primarily be seen as an effort to extend the operational potential of the commonly used expressions for energy and mo-

[8] E. J. Post, *Formal Structure of Electromagnetics* (John Wiley & Sons, Inc., New York, 1963).

[9] G. Hertz and H. Mende, Z. Physik **114**, 354 (1939).

mentum. To obtain a treatment of radiation forces on medium inhomogeneities it was necessary in the course of this investigation to modify certain normally accepted notions which stood in the way of a coherent development of the subject matter. They were:

(1) The commonly hidden assumption of homogeneity in the derivation of energy-momentum relations.
(2) The too-specialized definition of the wave momentum of radiation in terms of c.
(3) The impermissible imposition of a strict symmetry requirement on the energy-momentum tensor.

It should also be clear from the previous discussion that a considerable amount of detail and physical subtlety can be concealed in the transformation behavior of physical fields

It should be stressed that the present treatment of radiation forces is only a minor step towards a more coordinated description of energy-momentum relations. Further and deeper-going modifications are necessary.[10]

In this connection it is important to note that many physical distinctions developed in this article vanish under the customary substitution $ict=x_0$. The latter changes the indefinite metric into a definite metric thus obscuring how the physical identification of fields depends on the co- and contra-gradient behavior of transformation.

Finally it should be remarked that the description of nonuniformity requires the existence of a reference of uniformity. The use of local Cartesian frames is thus important for singling out the results of true physical inhomogeneities in the expression $\partial_{(\lambda)}\mathfrak{L}$ for the radiation force density.

[10] The physical meaning of the formal symmetrization procedures for the energy-momentum tensor for instance is a future topic to be considered.

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Zeeman and Coherence Effects in the He-Ne Laser*

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An account is given of further investigations of the Zeeman effect on the He-Ne laser transition at $\lambda=1.153\ \mu$, using both planar- and confocal-type resonators. For Zeeman level separations larger than the natural linewidths, the specific polarizations of the Zeeman transitions for the appropriate geometry are observed in the planar laser. Low-frequency splittings of axial resonances associated with anomalous dispersion effects occur under these conditions, the polarizations of these being linear, or circular, and orthogonal. At values of magnetic field such that the Zeeman levels overlap, coherence effects in the induced radiation are made evident by the disappearance of such low-frequency beats and by changes in these polarizations. This is considered using the theory of the depolarization of resonance radiation by magnetic fields, and also using time-dependent perturbation methods. For a symmetrical location of the axial resonance within the Doppler-broadened line, linear polarization is predicted for axial magnetic fields such that the states overlap, and some experimental verification is given. Related effects occur in the confocal laser where the Brewster angle windows determine the polarization. Here oscillations may be inhibited, or modulated by axial magnetic fields. Dips in the power output of this laser occur at smaller magnetic fields and are presently associated with interference effects when the Zeeman levels overlap. Some indications are given of coupling effects at Zeeman separations corresponding to the frequency interval between axial resonances.

1. INTRODUCTION

INVESTIGATIONS of the Zeeman effect have played a dominant role in the development of the quantum mechanical interpretation of atomic spectra. Wood and Ellett[1] in their pioneering work on the polarization of resonance radiation, mention the large depolarizing effects of small magnetic fields on the resonance radiation of mercury. The largest effect was obtained when the atoms were excited by incident π mode radiation and the polarization of the resonance radiation was observed using the σ transitions. For zero magnetic field the observed polarization was 90%, while for a field of 2 Oe the polarization was less than 1%. The polarization of the resonance radiation is thus linear, and in the same direction as that of the incident radiation for zero magnetic field, but decreases continuously with increasing magnetic field, becoming elliptical with a rotation of the plane of maximum polarization. Similar variations in the polarization occur for other orientations of the magnetic field and incident polarization, the phenomena now being called the Hanle effect,[2] who made a thorough investigation of it.

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[1] R. W. Wood and A. Ellett, Phys. Rev. 24, 243 (1924).

[2] A. C. G. Mitchell and M. W. Zemansky, *Resonance Radiation and Excited Atoms* (Cambridge University Press, New York, 1934), pp. 258–317.